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PERTURBATION SOLUTIONS OF PLANAR DIFFUSION-CONTROLLED MOVING-BOUNDARY PROBLEMS

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Abstract—A new perturbation method is developed for the problems of solidification of liquid in a plane coordinate system. The method consists of (1) immobilizing the moving boundary by Landau's transformation, (2) replacing the time variable by the moving interface $x_f(\tau)$, (3) applying the regular parameter perturbation technique. A quasi-steady state solution is shown to be the zero-order approximation. The perturbation solution for the planar solidification of a saturated liquid with constant wall temperature is shown to be identical with the exact solution. Comparison of the perturbation solutions for the solidification of a flowing warm liquid on a cooled flat plate with the experimental result of Siegel and Savino [8] is also given.

NOMENCLATURE

a, thickness of wall on which frozen layer is forming;
Bi, Biot number, $h_0 X_s/k$;
c_p, specific heat at constant pressure of frozen layer;
F_1, F_2, formulas, defined by equation (34);
h_c, convective heat-transfer coefficient between coolant and wall;
h_0, overall convective heat-transfer coefficient, equation (23);
h_1, convective heat-transfer coefficient between frozen layer and liquid;
k, thermal conductivity of frozen layer;
k_w, thermal conductivity of wall on which frozen layer is forming;
L, latent heat of fusion of freezing material;
T, temperature distribution in frozen layer;
T_0, temperature of coolant;
T_f, freezing temperature;
T_1, temperature of flowing liquid;
t, time;
U, dimensionless temperature, $(T_f - T)/(T_f - T_0)$;
U_i, coefficient of ε^i in the asymptotic expansion of U;
X, position coordinate in frozen layer measured from wall;
X_f, thickness of frozen layer;
X_s, thickness of frozen layer at steady state, equation (24), or characteristic length, equation (2);
x, dimensionless position coordinate, X/X_s;
$x_f(\tau)$, dimensionless thickness of frozen layer, X_f/X_s.

Greek symbols

δ, normalized distance;
ε, Stefan number, $c_p(T_f - T_0)/L$;
λ, growth constant, equation (5);
ρ, density of frozen layer;
τ, Fourier number, dimensionless time, $kt/(\rho c_p X_s^2)$;
τ_i, coefficient of ε^i in the asymptotic expansion of $\varepsilon\tau$.

INTRODUCTION

BANKOFF [1, 2] reviewed various analytical and approximation methods for the diffusion-controlled moving boundary problems. Muehlbauer and Sunderland [3] also reviewed heat conduction with freezing or melting. Since then, new results have been obtained by many investigators. Tao [4, 5] obtained the numerical solutions of freezing saturated liquid in cylinders and spheres by a finite difference method. Numerical methods have been used by Lock *et al.* [6] and Cho and Sunderland [7] by first using Landau transformation [1] to immobilize the boundary. The analytical iterative solutions for the freezing in plane coordinate systems obtained by Savino and Siegel [8, 9] have been extended to cylindrical and spherical coordinate systems by Shih and Tsay [10], Shih and Chou [11], and Theofanous and Lim [12].

Application of perturbation methods to diffusion-controlled moving boundary problems needs special attention. Duda and Vrentas [13, 14] compared the exact solution of the growth of a plane surface of vapor into an infinite liquid phase of binary system with a perturbation solution and found that the zero-order approximate solution is the quasi-stationary solution. They also discussed the importance and usefulness of the perturbation method. Pedroso and Domoto [15–17] encountered difficulty in the application of a perturbation method to inward spherical solidification. Lock [18] and Jiji [19] applied perturbation methods to freezing on a flat plane and outside a cylinder, respectively.

In this report, a new perturbation method is developed and is tested for the flat-plate freezing problem.

This new method consists of three major steps: (a) applying Landau's transformation [1] to immobilize the moving boundary, (b) changing the time variable from τ to $x_f(\tau)$, which is the position of the moving interface, and (c) using the regular parameter perturbation method [20, 21]. The first two steps are the key points of the new method.

First, the freezing of liquid on a flat plate with constant wall temperature is analyzed. This problem has an exact solution. The perturbation solution is shown to be identical with the exact solution. A perturbation method is then applied to the freezing of liquid on a flat plate with convective boundary conditions. This problem does not have an exact solution. Comparison of the perturbation solution with the experimental result of Siegel and Savino [8] is discussed. Analysis of moving boundary problems of cylindrical and spherical coordinate systems will be presented in other reports.

FREEZING ON A FLAT PLATE WITH CONSTANT WALL TEMPERATURE

The system

Assuming constant physical properties of solid and negligible volume change the solidification of a liquid initially at the freezing temperature T_f can be described by the following equations for the plane coordinate system:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial X^2}, \qquad 0 \leqslant X \leqslant X_f(t) \tag{1a}$$

$$T(0, t) = T_0 \tag{1b}$$

$$T[X_f(t), t] = T_f \tag{1c}$$

$$\rho L \frac{\mathrm{d}X_f}{\mathrm{d}t} = k\left.\frac{\partial T}{\partial X}\right|_{X = X_f} \tag{1d}$$

$$X_f(0) = 0. \tag{1e}$$

Here the wall temperature is assumed to be constant at T_0. Defining the following dimensionless variables by the introduction of a reference distance X_s,

$$\begin{aligned} U &= \frac{T_f - T}{T_f - T_0} \\ \tau &= \frac{kt}{\rho c_p X_s^2} \\ x &= \frac{X}{X_s} \\ x_f(\tau) &= \frac{X_f}{X_s} \\ \varepsilon &= \frac{c_p(T_f - T_0)}{L} \end{aligned} \tag{2}$$

equations (1a–e) become:

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial X^2}, \qquad 0 \leqslant x \leqslant x_f(\tau) \tag{3a}$$

$$U(0, \tau) = 1 \tag{3b}$$

$$U[x_f(\tau), \tau] = 0 \tag{3c}$$

$$\frac{\mathrm{d}x_f}{\mathrm{d}\tau} = -\varepsilon\left.\frac{\partial U}{\partial x}\right|_{x = x_f(\tau)} \tag{3d}$$

$$x_f(0) = 0. \tag{3e}$$

Exact solution

An exact solution is available for the system of equations (3a–e) [22]:

$$U(x, \tau) = 1 - \mathrm{erf}(x/2\sqrt{\tau})/\mathrm{erf}(\lambda) \tag{4}$$

with

$$(\sqrt{\pi})\lambda \exp(\lambda^2)\mathrm{erf}(\lambda) = \varepsilon \tag{5}$$

and

$$x_f(\tau) = 2\lambda\sqrt{\tau}. \tag{6}$$

Expanding the function $\exp(\lambda^2)\mathrm{erf}(\lambda)$ in terms of λ, equation (5) becomes [23]:

$$\varepsilon = 2(\lambda^2 + \frac{2}{1.3}\lambda^4 + \frac{2^2}{1.3.5}\lambda^6 + \frac{2^3}{1.3.5.7}\lambda^8 + \ldots. \tag{7}$$

Reversion of the power series of λ of equation (7) into the power series of ε gives [24]:

$$2\lambda^2 = \varepsilon - \tfrac{1}{3}\varepsilon^2 + \tfrac{7}{45}\varepsilon^3 - \tfrac{79}{945}\varepsilon^4 + \ldots. \tag{8}$$

Letting

$$\delta = \frac{x}{x_f(\tau)}$$

equation (4) becomes [23]:

$$U(\delta, \tau) = 1 - \frac{\mathrm{erf}(\lambda\delta)}{\mathrm{erf}(\lambda)}$$

$$= 1 - \frac{\lambda\delta - \dfrac{(\lambda\delta)^3}{3.1!} + \dfrac{(\lambda\delta)^5}{5.2!} - \dfrac{(\lambda\delta)^7}{7.3!} + \ldots}{\lambda - \dfrac{\lambda^3}{3.1!} + \dfrac{\lambda^5}{5.2!} - \dfrac{\lambda^7}{7.3!} + \ldots} \tag{9}$$

or

$$U(\delta, \tau) = 1 - \delta + \frac{2\lambda^2}{6}(\delta^3 - \delta) - \frac{(2\lambda^2)^2}{360}(9\delta^5 - 10\delta^3 + \delta) + \frac{(2\lambda^2)^3}{15\,120}(45\delta^7 - 63\delta^5 + 7\delta^3 + 11\delta) - \ldots. \tag{10}$$

The expression of $U(\delta, \tau)$ in the power series of ε can be obtained by the substitution of equation (8) into equation (10):

$$U(\delta, \tau) = (1 - \delta) + \frac{\varepsilon}{6}(\delta^3 - \delta) + \varepsilon^2\left(-\frac{\delta^5}{40} - \frac{\delta^3}{36} + \frac{19\delta}{360}\right) + \varepsilon^3\left(\frac{\delta^7}{336} + \frac{8^5}{80} + \frac{17\delta^3}{2160} - \frac{353\delta}{15\,120}\right). \tag{11}$$

Substitution of equation (11) into equations (3d–e), the solution of $\tau(x_f)$ is

$$\varepsilon\tau = \tau_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2 + \varepsilon^3\tau_3 + \ldots \tag{12}$$

where

$$\begin{aligned} \tau_0 &= x_f^2/2 \\ \tau_1 &= 1/3 . x_f^2/2 \\ \tau_2 &= -2/45 . x_f^2/2 \\ \tau_3 &= -142/945 . x_f^2/2. \end{aligned}$$

Inversion of equation (12) gives $x_f^2(\tau)$ as a power series of ε:

$$\begin{aligned} x_f^2(\tau) &= 2\lambda^2 . 2\tau \\ &= 2\tau(\varepsilon - \tfrac{1}{3}\varepsilon^2 + \tfrac{7}{45}\varepsilon^3 - \tfrac{79}{945}\varepsilon^4 + \ldots). \end{aligned} \tag{13}$$

Equations (11) and (12) are the exact solution which will be compared with the perturbation solution obtained in the next section, represented by a power series of the parameter ε.

Perturbation solution

In equations (3a–e), the independent variables of U are τ and x, $U = U(x, \tau)$. Changing the position coordinate x to δ gives $U = U(\delta, \tau)$. This is known as Landau's transformation. Landau's transformation makes the nonlinearity due to moving interface explicit. Furthermore, changing the time variable τ to $x_f(\tau)$, gives $U = U(\delta, x_f)$. The latter transformation is possible when x_f is a monotonic function of τ. Many practical applications belong to this case.

In the case that $U = U(\delta, x_f)$, equations (3a–e) become

$$\varepsilon\left(\delta\frac{\partial U}{\partial \delta} - x_f\frac{\partial U}{\partial x_f}\right)\left(\left.\frac{\partial U}{\partial \delta}\right|_{\delta=1}\right) = \frac{\partial^2 U}{\partial \delta^2} \tag{14a}$$

$$U(0, x_f) = 1 \tag{14b}$$

$$U(1, x_f) = 0 \tag{14c}$$

$$\frac{d\tau}{dx_f} = -\frac{x_f}{\varepsilon\left.\dfrac{\partial U}{\partial \delta}\right|_{\delta=1}} \tag{14d}$$

$$\tau(x_f) = 0, \quad \text{at} \quad x_f = 0. \tag{14e}$$

The technique of regular parameter perturbation is then used to analyze equations (14a–e). Asymptotic expansion of $U(\delta, x_f)$ into the power series of the parameter ε yields

$$U(\delta, x_f) = U_0(\delta, x_f) + \varepsilon U_1(\delta, x_f) + \varepsilon^2 U_2(\delta, x_f) + \ldots. \tag{15}$$

Substituting equation (15) into equation (14a) and equating the coefficients of equal powers of ε yield

$$\frac{\partial^2 U_0}{\partial \delta^2} = 0 \tag{16}$$

$$\frac{\partial^2 U_1}{\partial \delta^2} = \left(\delta\frac{\partial U_0}{\partial \delta} - x_f\frac{\partial U_0}{\partial x_f}\right)\left(\left.\frac{\partial U_0}{\partial \delta}\right|_{\delta=1}\right) \tag{17}$$

$$\frac{\partial^2 U_2}{\partial \delta^2} = \left(\delta\frac{\partial U_0}{\partial \delta} - x_f\frac{\partial U_0}{\partial x_f}\right)\left.\frac{\partial U_1}{\partial \delta}\right|_{\delta=1} + \left(\delta\frac{\partial U_1}{\partial \delta} - x_f\frac{\partial U_1}{\partial x_f}\right)\left(\left.\frac{\partial U_0}{\partial \delta}\right|_{\delta=1}\right) \tag{18}$$

$$\frac{\partial^2 U_i}{\partial \delta^2} = \sum_{j=0}^{i-1}\left(\delta\frac{\partial U_j}{\partial \delta} - x_f\frac{\partial U_j}{\partial x_f}\right)\left(\left.\frac{\partial U_{i-j}}{\partial \delta}\right|_{\delta=1}\right). \tag{19}$$

Substitution of equation (15) into equations (14b) and (14c) gives, respectively, the boundary conditions for $U_i(\delta, x_f)$

$$\begin{aligned} &U_0(0, x_f) = 1 \\ &U_i(0, x_f) = 0, \; i = 1, 2, \ldots \\ &U_i(1, x_f) = 0, \; i = 0, 1, 2, \ldots. \end{aligned} \tag{20}$$

The expressions of the first four terms, $U_i(\delta, x_f)$, $i = 0, 1, 2, 3$, are obtained from equations (16–20):

$$\begin{aligned} U_0(\delta, x_f) &= 1 - \delta \\ U_1(\delta, x_f) &= \tfrac{1}{6}(\delta^3 - \delta) \\ U_2(\delta, x_f) &= -\frac{\delta^5}{40} - \frac{\delta^3}{60} + \frac{19\delta}{360} \\ U_3(\delta, x_f) &= \frac{\delta^7}{336} + \frac{\delta^5}{80} + \frac{17\delta^3}{2160} - \frac{353\delta}{15\,120}. \end{aligned} \tag{21}$$

The zero-order solution $U_0(\delta, x_f) = 1 - \delta$ is the quasi-steady state solution, which can be obtained by neglecting the time derivative of equation (3a). The quasi-steady state solution is also known as the pseudo-steady state solution. When ε is sufficiently small, higher order terms can be neglected. This gives a formal proof of the quasi-steady state solution. Notice that Duda and Vrentas [13] found that the zero-order solution is the quasi-stationary solution of a class of diffusion-controlled bubble growth problems.

The perturbation solution of the temperature profile of equations (15) and (21) is identical with the exact solution given by equation (11). The perturbation solution of the moving interface obtained from equations (14d–e) as well as equations (15) and (21) is also identical with the exact solution given by equation (12).

SOLIDIFICATION OF A FLOWING WARM LIQUID ON COOLED FLAT PLATE

The system

The solidification of a warm liquid flowing over a flat plate which is cooled below was studied experimentally and analytically by Siegel and Savino [8]. This problem is studied here by the use of the perturbation method. Referring to Fig. 1 and neglecting the

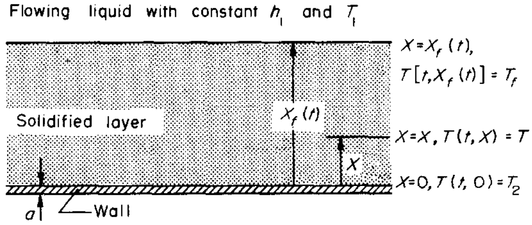

FIG. 1. Freezing of liquid flowing over a chilled flat plate.

heat capacity of the chilled wall, the problem can be described as follows:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial X^2}, \qquad 0 \leqslant X \leqslant X_f(t) \tag{22a}$$

$$k\left.\frac{\partial T}{\partial X}\right|_{X=0} = h_0[T(0, t) - T_0] \tag{22b}$$

$$T[X_f(t), t] = T_f \tag{22c}$$

$$\rho L\frac{dX_f}{dt} = k\left.\frac{\partial T}{\partial X}\right|_{X=X_f(t)} - h_1(T_1 - T_f) \tag{22d}$$

$$X_f(0) = 0. \tag{22e}$$

Here h_0 is the overall convective heat-transfer coefficient from the coolant to the chilled wall at $X = 0$:

$$\frac{1}{h_0} = \frac{1}{h_c} + \frac{a}{k_w}. \tag{23}$$

The thickness of the frozen layer approaches a steady-state value, X_s, which can be obtained by heat balance [8]:

$$X_s = k(T_f - T_0)/h_1(T_1 - T_0) - k(a/k_w + 1/h_c). \tag{24}$$

Defining dimensionless variables and parameters by equations (2) and Biot number, Bi, as

$$Bi = h_0 X_s/k. \tag{25}$$

Equations (22a–e) become:

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 \leqslant x \leqslant x_f(\tau) \tag{26a}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = Bi[U(0,\tau) - 1] \tag{26b}$$

$$U(x_f, \tau) = 0 \tag{26c}$$

$$\frac{dx_f}{d\tau} = -\varepsilon\left(\left.\frac{\partial U}{\partial x}\right|_{x=x_f} + \frac{Bi}{1+Bi}\right) \tag{26d}$$

$$x_f(0) = 0. \tag{26e}$$

Transformation of independent variables

No exact solution exists for this system. Define $\delta = x/x_f(\tau)$. To facilitate the perturbation approach, the independent variables τ and x are changed to $x_f(\tau)$ and δ, respectively. Hence $U = U[\delta, x_f(\tau)]$ and equations (26a–e) become:

$$\varepsilon\left(\delta\frac{\partial U}{\partial \delta} - x_f\frac{\partial U}{\partial x_f}\right)\left(\left.\frac{\partial U}{\partial \delta}\right|_{\delta=1} + \frac{Bix_f}{1+Bi}\right) = \frac{\partial^2 U}{\partial \delta^2}, \qquad 0 \leqslant \delta \leqslant 1 \tag{27a}$$

$$\left.\frac{\partial U}{\partial \delta}\right|_{\delta=0} = Bix_f[U(0,x_f) - 1] \tag{27b}$$

$$U(1, x_f) = 0 \tag{27c}$$

$$\varepsilon\frac{d\tau}{dx_f} = -\left(\frac{1}{x_f}\left.\frac{\partial U}{\partial \delta}\right|_{\delta=1} + \frac{Bi}{1+Bi}\right)^{-1} \tag{27d}$$

$$\tau(x_f) = 0, \qquad \text{at} \qquad x_f = 0. \tag{27e}$$

Perturbation solution

The asymptotic expansion of $U(\delta, x_f)$ in the power series of ε is given by equation (15). The first three terms of temperature distribution as in the following are obtained by substituting equation (15) into equation (27a) and by equating the coefficients of the same powers of ε

$$\frac{\partial^2 U_0}{\partial \delta^2} = 0 \tag{28}$$

$$\frac{\partial^2 U_1}{\partial \delta^2} = \left(\delta\frac{\partial U_0}{\partial \delta} - x_f\frac{\partial U_0}{\partial x_f}\right)\left(\left.\frac{\partial U_0}{\partial \delta}\right|_{\delta=1} + \frac{Bix_f}{1+Bi}\right) \tag{29}$$

$$\frac{\partial^2 U_2}{\partial \delta^2} = \left(\delta\frac{\partial U_0}{\partial \delta} - x_f\frac{\partial U_0}{\partial x_f}\right)\left(\left.\frac{\partial U_1}{\partial \delta}\right|_{\delta=1}\right) + \left(\delta\frac{\partial U_1}{\partial \delta} - x_f\frac{\partial U_1}{\partial x_f}\right)\left(\left.\frac{\partial U_0}{\partial \delta}\right|_{\delta=1} + \frac{Bix_f}{1+Bix_f}\right). \tag{30}$$

The boundary conditions needed for the linear equations of (28), (29), and (30) are obtained from equations (15), (27b) and (27c):

$$\begin{aligned}
\left.\frac{\partial U_0}{\partial \delta}\right|_{\delta=0} &= Bix_f[U_0(0,x_f) - 1] \\
\left.\frac{\partial U_1}{\partial \delta}\right|_{\delta=0} &= Bix_f U_1(0,x_f) \\
\left.\frac{\partial U_2}{\partial \delta}\right|_{\delta=0} &= Bix_f U_2(0,x_f) \\
U_0(1,x_f) &= 0 \\
U_1(1,x_f) &= 0 \\
U_2(1,x_f) &= 0.
\end{aligned} \tag{31}$$

The solutions of $U_0(\delta, x_f)$, $U_1(\delta, x_f)$ and $U_2(\delta, x_f)$ from equations (28–30) are:

$$U_0(\delta, x_f) = \frac{Bix_f}{1+Bix_f}(1-\delta) \tag{32}$$

$$U_1(\delta, x_f) = \frac{Bi^3 x_f^2(1-x_f)}{6(1+Bi)(1+Bix_f)^4} \times [(1+Bix_f)(3+Bix_f\delta)\delta^2 - (3+Bix_f)(1+Bix_f\delta)] \tag{33}$$

$$U_2(\delta, x_f) = \frac{Bi^4 x_f^3(1-x_f)}{360(1+Bi)^2(1+Bix_f)^7} \times \{(1+Bix_f)[10(3+Bix_f\delta)\delta^2 F_2 + 3(5+Bix_f\delta)\delta^4]F_1 - (1+Bix_f\delta) + [3(5+Bix_f)F_1 - 10(3+Bix_f)F_2]\} \tag{34}$$

where

$$F_1 = Bix_f(1+Bix_f)(1+3Bi-2Bix_f)$$
$$F_2 = 6(1+2Bi) - 3(1-Bi)Bix_f + (1+Bi)Bi^2x_f^2.$$

Notice that $U_0(\delta, x_f)$ of equation (32) is the quasi-steady state solution, which can be obtained from equations (26a–c) by neglecting the time derivative of equation (26a).

Position of moving interface

The position of the moving interface is calculated from equations (27d), (27e) and (25).

$$\varepsilon\frac{d\tau}{dx_f} = -\frac{x_f}{\dfrac{Bix_f}{1+Bi} + \displaystyle\sum_{i=0}^{\infty}\varepsilon^i\left.\frac{\partial U_i}{\partial \delta}\right|_{\delta=1}}. \tag{35}$$

Integration of equation (35) from 0 to x_f yields

$$\varepsilon\tau = -\int_0^{x_f}\frac{x_f\,dx_f}{\dfrac{Bix_f}{1+Bi} + \displaystyle\sum_{i=0}^{\infty}\varepsilon^i\left.\frac{\partial U_i}{\partial \delta}\right|_{\delta=1}}. \tag{36}$$

Letting

$$\varepsilon\tau = \tau_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2 + \ldots.$$

Equations (32–34) and (36) give

$$\tau_0 = -\frac{1+Bi}{Bi}\left[x_f + \frac{1+Bi}{Bi}\ln(1-x_f)\right] \tag{37}$$

$$\tau_1 = \frac{-1-Bi}{3Bi}\left[x_f + \frac{x_f}{(1+Bi)(1+Bix_f)} + \frac{3+3Bi+Bi^2}{(1+Bi)^2} \times \ln(1-x_f) + \frac{\ln(1+Bix_f)}{Bi(1+Bi)^2}\right] \quad (38)$$

$$\tau_2 = \int_0^{x_f} \frac{x_f^2[10(1+Bi)(3+3Bix_f+Bi^2x_f^2)^2 - H]}{90(1-x_f)(1+Bix_f)^5}\,\mathrm{d}x_f \quad (39)$$

where

$$H = 3(5+5Bix_f+Bi^2x_f^2)F_1 + 5(3+3Bix_f+Bi^2x_f^2)F_2.$$

Here F_1 and F_2 are given in equation (34).

Notice that $\varepsilon\tau = \tau_0$ is the quasi-steady state solution. Table 1 gives the values of τ_0, τ_1 and τ_2 for different Biot number.

Table 1. τ_0, τ_1 and τ_2 for planar solidification with a warm liquid flowing over a chilled flat plate

x_f	τ_0	τ_1	τ_2
		$Bi = 0·1$	
0·00	0·0000	0·000	0·0000
0·05	5·6565	0·01418	$-4·051 \times 10^{-6}$
0·10	11·649	0·05857	$-3·146 \times 10^{-5}$
0·15	18·015	0·1363	$-1·029 \times 10^{-4}$
0·20	24·800	0·2512	$-2·361 \times 10^{-4}$
0·25	32·060	0·4076	$-4·453 \times 10^{-4}$
0·30	39·858	0·6110	$-7·409 \times 10^{-4}$
0·35	48·275	0·8679	−0·001128
0·40	57·410	1·1865	−0·001610
0·45	67·388	1·5772	−0·002178
0·50	78·371	2·0534	−0·002819
0·55	90·569	2·6324	−0·003506
0·60	104·27	3·3382	−0·004197
0·65	119·80	4·2039	−0·004823
0·70	137·98	5·2786	−0·005277
0·75	159·49	6·6378	−0·005382
0·80	185·94	8·4085	−0·004833
0·85	220·20	10·829	−0·003047
0·90	268·71	14·432	0·001304
0·95	352·03	20·923	0·01248

Table 1. (Continued)

x_f	τ_0	τ_1	τ_2
		$Bi = 0·5$	
0·00	0·0000	0·0000	0·0000
0·05	0·3116	0·003816	$-1·909 \times 10^{-5}$
0·10	0·6482	0·01557	$-1·400 \times 10^{-4}$
0·15	1·0127	0·03579	$-4·335 \times 10^{-4}$
0·20	1·4083	0·06515	$-9·422 \times 10^{-4}$
0·25	1·8391	0·1045	−0·001685
0·30	2·3101	0·1548	−0·002663
0·35	2·8270	0·2175	−0·003859
0·40	3·3974	0·2941	−0·005237
0·45	4·0305	0·3868	−0·006746
0·50	4·7383	0·4983	−0·008313
0·55	5·537	0·6322	−0·009839
0·60	6·447	0·7935	−0·01119
0·65	7·498	0·9891	−0·01218
0·70	8·736	1·2293	−0·01253
0·75	10·227	1·5299	−0·01185
0·80	12·085	1·9177	−0·009447
0·85	14·524	2·4426	−0·004044
0·90	18·023	3·2169	0·007185
0·95	24·112	4·5995	0·03325
		$Bi = 1·0$	
0·00	0·0000	0·0000	0·0000
0·05	0·1052	0·002504	$-3·554 \times 10^{-5}$
0·10	0·2214	0·01007	$-2·441 \times 10^{-4}$
0·15	0·3501	0·02283	$-7·112 \times 10^{-4}$
0·20	0·4926	0·04106	−0·001461
0·25	0·6507	0·06510	−0·002482
0·30	0·8267	0·09547	−0·003737
0·35	1·0231	0·1328	−0·005175
0·40	1·2433	0·1780	−0·006730
0·45	1·4913	0·2321	−0·008324
0·50	1·7726	0·2966	−0·009866
0·55	2·0940	0·3736	−0·01124
0·60	2·4652	0·4657	−0·01230
0·65	2·8993	0·5767	−0·01287
0·70	3·4159	0·7123	−0·01267
0·75	4·0452	0·8812	−0·01129
0·80	4·8378	1·0932	−0·008088
0·85	5·8885	1·3910	−0·001812
0·90	7·4103	1·8215	0·01027
0·95	10·083	2·5880	0·03692

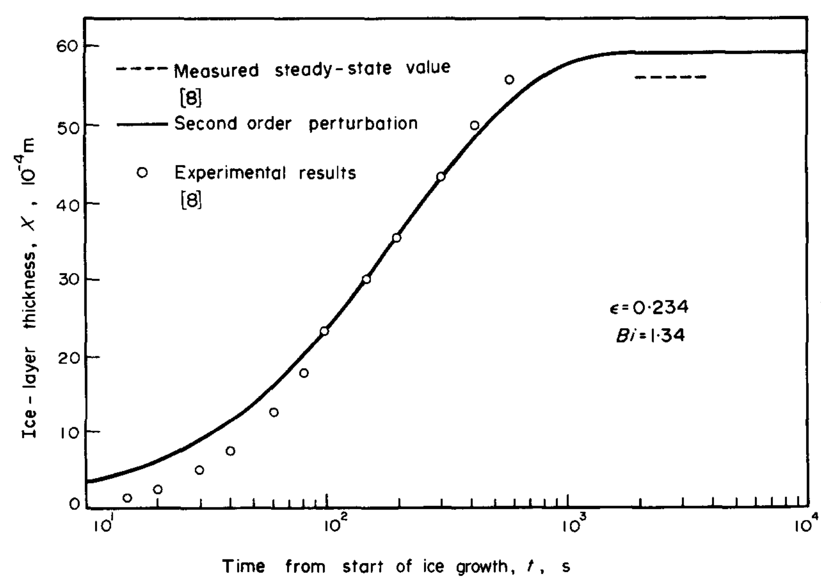

FIG. 2. Comparison of experimental result [8] with second-order perturbation solution.

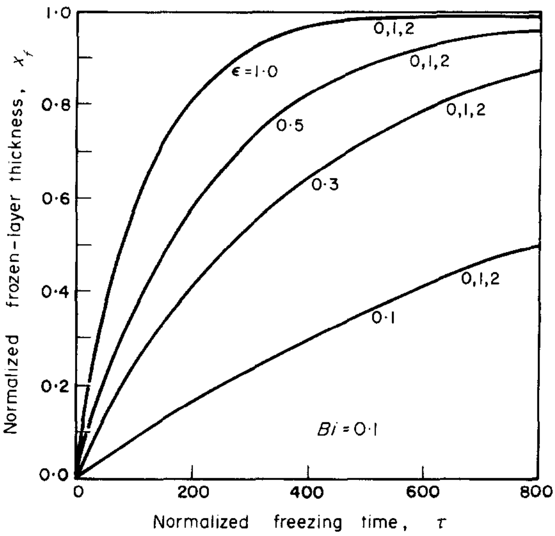

FIG. 3. Convergence of perturbation solution, $Bi = 0{\cdot}1$. 0—$\tau = \tau_0/\varepsilon$; 1—$\tau = (\tau_0+\varepsilon\tau_1)/\varepsilon$; 2—$\tau = (\tau_0+\varepsilon\tau_1+\varepsilon^2\tau_2)/\varepsilon$.

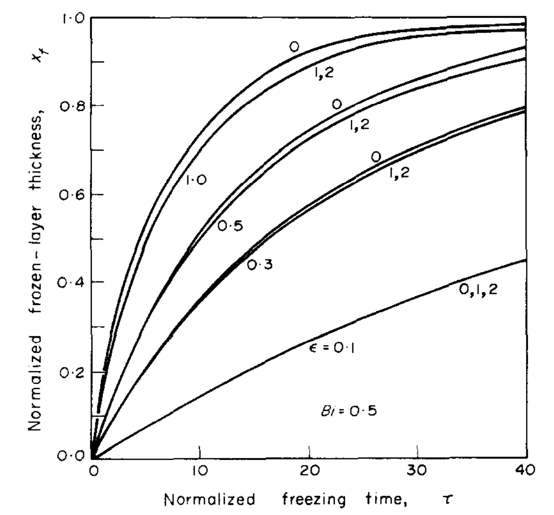

FIG. 4. Convergence of perturbation solution, $Bi = 0{\cdot}5$. 0—$\tau = \tau_0/\varepsilon$; 1—$\tau = (\tau_0+\varepsilon\tau_1)/\varepsilon$; 2—$\tau = (\tau_0+\varepsilon\tau_1+\varepsilon^2\tau_2)/\varepsilon$.

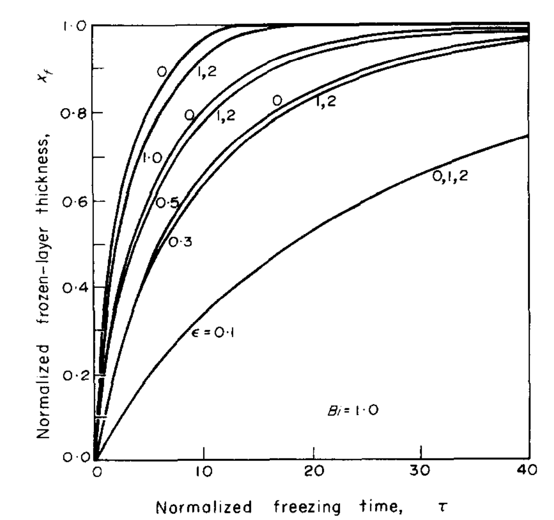

FIG. 5. Convergence of perturbation solution, $Bi = 1{\cdot}0$. 0—$\tau = \tau_0/\varepsilon$; 1—$\tau = (\tau_0+\varepsilon\tau_1)/\varepsilon$; 2—$\tau = (\tau_0+\varepsilon\tau_1+\varepsilon^2\tau_2)/\varepsilon$.

Comparison of the perturbation solution with the experimental results of Siegel and Savino [8] is illustrated in Fig. 2. Since $\varepsilon = 0{\cdot}234$ is rather small, good agreement is expected. Figures 3–5 show the convergence of the perturbation solution. For the values of the Biot number used the zero-order approximation is good for $\varepsilon < 0{\cdot}1$ and first-order approximation is good for $\varepsilon < 1$. Better convergence is obtained for small Biot number.

DISCUSSION AND CONCLUSIONS

Perturbation solutions are obtained for moving-boundary problems with plane coordinate. Two special transformations of the independent variables are used. The first transformation is Landau's transformation to immobilize the boundary condition. Then the time variable τ is replaced by the moving interface $x_f(\tau)$, provided $x_f(\tau)$ is a monotonic function of τ. A regular parameter perturbation technique is used in a straightforward manner. For freezing of a saturated liquid on a wall of constant temperature, the exact solution is identical with the perturbation solution.

This report also formally proves that the quasi-steady state solution is the zero-order approximate solution for the problems considered.

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